Simulation of non equilibrium plasmas with a numerical noise reduced particle in cell method.

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Abstract. An innovative Particle In Cell method, aimed at reducing the numerical noise characterising classical PIC schemes, has been investigated in the framework of ionospheric plasma simulations. The numerical model is composed of a kinetic equation for the plasma coupled with the Maxwell-Faraday law driving the evolution in the magnetic field.

INTRODUCTION

In this paper we investigate the evolution of the ionospheric plasma and more precisely, the interaction of plasma bubbles with the earth magnetic field. The plasma composing these bubbles can be assumed quasi-neutral but not in a thermodynamical equilibrium and thus requires a kinetic description. In these first stages of the study the collision processes, although very important in the ionosphere context [see 1], are disregarded and postponed to a future work. We propose a simplified model consisting of a Vlasov equation for the plasma coupled to the Maxwell-Faraday equation for the magnetic field evolution, where the electric field is provided by a generalised Ohm's law. It is derived under the same assumptions as the Magneto-Hydro-Dynamic model but with a kinetic description of the plasma. The numerical method relies on a moment guided method introduced in [5] which aims at reducing the numerical noise of classical particle method. It has proven to be very efficient in the context of collisional rarefied gas dynamics and is investigated here in a different framework. One dimensional numerical experiments are proposed to give a first view of the numerical method efficiency.

A MODEL FOR THE EVOLUTION OF A IONOSPHERIC PLASMA IN A NON THERMODYNAMICAL EQUILIBRIUM

A kinetic description of the ionospheric plasma

The system relies on a kinetic description for the ions and a fluid one for the electrons coupled to the Maxwell's equations for the electromagnetic field evolution. The equations are detailed with dimensionless quantities obtained thanks to typical space and velocity scales $\overline{x}$ and $\overline{u}$ defining the time scale $\overline{t} = \overline{x}/\overline{u}$. We denote by $\overline{E}$, $\overline{B}$, $\overline{T}$ and $\overline{n}$ the typical values for the electric and magnetic fields as well as the plasma temperature and density, we introduce $\overline{f} = \overline{n}/(\overline{u}^3)$ the typical value for the distribution function. Using these scales to define dimensionless variables yields

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \eta \left(E + \alpha\beta v \times B\right) \cdot \nabla_v f = 0\,. \tag{1}$$

The electro-magnetic field (E,B) is a function of x and t, its evolution being driven by the Maxwell's equations

$$\frac{\alpha}{\beta}\frac{\partial E}{\partial t}-\nabla_x\times B=0=-\frac{\alpha}{\beta\eta\lambda^2}j\,, \tag{2}$$

$$\partial_t B+\nabla_x\times E=0\,, \tag{3}$$

$$\eta\lambda^2\nabla_x\cdot E=\rho\,, \tag{4}$$

$$\nabla_x\cdot B=0\,. \tag{5}$$

In these equations ρ and j are respectively the charge and current densities as defined by $\rho=n_e-n_i$, $j=n_iu_i-n_eu_e$, where the electronic density and velocity obey

$$\partial_t n_e+\nabla_x\cdot(n_eu_e)=0\,, \tag{6}$$

$$\varepsilon^2\left(\partial_t(n_eu_e)+\nabla_x\cdot(n_eu_e\otimes u_e)\right)+\nabla_x p_e(n_e)=-\eta n_e\left(E+\alpha\beta u_e\times B\right)\,. \tag{7}$$

The ionic density and momentum are provided by $n=\int f\,dv$, $nu_i=\int f\,v\,dv$. This system relies on the following dimensionless parameters

- ε $=m_e/m_i$ the ratio of the electronic and ionic masses;
- β $=c\overline{B}/\overline{E}$ where c is speed of light;
- α $=\overline{u}/c$ the ratio of the typical macroscopic velocity to the speed of light;
- λ^2 $=\varepsilon_0 k_B\overline{T}/(e^2\overline{nx}^2)$ is the dimensionless squared Debye length, ε_0 being the vacuum permittivity, e the elementary charge;
- η $=e\overline{E}\overline{x}/m_i\overline{u}^2$ is the ratio of the momentum transport term and the Lorentz force.

Scaling relations

First we assume an entire Lorentz force comparable to the pressure term for both electrons and ions, which means $\eta=1$ as well as $\alpha\beta=1$. The electron inertia is then disregarded and the plasma is assumed quasi-neutral which translates into $\varepsilon\to 0$ and $\lambda\to 0$. The latter limit, thanks to the Gauss law (4), provides $n_e=n_i=n$, the former one transforms the electronic momentum equation (7) in the so-called generalised Ohm's law :

$$\nabla_x p_e=-n\left(E+u\times B\right)+j\times B\,. \tag{8}$$

In this expression $u=u_i$ is the plasma velocity. Finally the current caused by the particle motion is assumed large enough to produce changes in the magnetic field. This last statement yields $\alpha/(\beta\eta\lambda^2)=1$ which, using all the above hypotheses, gives $\alpha/\beta=\lambda^2$: the displacement current vanishes in Ampere's equation which degenerates into

$$\nabla_x\times B=j\,. \tag{9}$$

A simplified model problem for the ionospheric plasma interactions with the ambient magnetic field

The model obtained with these scaling relations and expressed with dimensional quantities reads as a Vlasov equation coupled with the Maxwell-Faraday law

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f+\frac{e}{m_i}\left(E+v\times B\right)\cdot\nabla_v f=0\,, \tag{10}$$

$$\frac{\partial B}{\partial t}+\nabla_x\times E=0\,, \tag{11}$$

where the magnetic field is divergence free, $\nabla_x\cdot B=0$ and the electric field is defined by (8) as,

$$E=-\big(u-\nabla_x\times B/(\mu_0\,en)\big)\times B\,, \tag{12}$$

μ_0 being the vacuum permeability. Note that, for sake of simplicity we have omitted the electronic pressure term in the generalised Ohm's law. These equations can be regarded as a kinetic extension of the Hall-MHD system. This point will be detailed further in the sequel.

NUMERICAL METHODS

The “moment guided method”

The moment guided method has been introduced in [5] in the context of rarefied gas dynamics described by the Boltzmann equation. It is aimed at reducing the numerical noise characteristic of particle methods widely used to discretize kinetic equations. The idea is to decompose the distribution function as a Maxwellian corrected with an additional function g: $f = \mathscr{M}_{n,u,T} + g$. In this decomposition the Maxwellian $\mathscr{M}_{n,u,T}$ shares the same first three moments with the distribution function $n(x,t) = \langle f \rangle$, $u(x,t) = \frac{1}{n}\langle v\, f \rangle$, $T(x,t) = \frac{\partial}{\partial n}\langle |v-u|^2 f \rangle$, where $\langle f \rangle = \int f(x,v,t)\, dv$. Inserting this decomposition in the Vlasov equation and computing the first moments yields the *moment model*

$$
\begin{aligned}
&\frac{\partial n}{\partial t} + \nabla_x \cdot (n u) = 0\,,\\
&m\frac{\partial nu}{\partial t} + \nabla_x \cdot \left(mnu \otimes u - \frac{1}{\mu_0} B \otimes B\right) + \nabla_x p_{\mathrm{TOT}} = -m\nabla_x \cdot \left\langle v \otimes vg \right\rangle,\\
&\frac{\partial W_{\mathrm{TOT}}}{\partial t} + \nabla \cdot \left(W_{\mathrm{TOT}} u + p_{\mathrm{TOT}} u - \frac{1}{\mu_0}(B\cdot u)B\right) = -\frac{m}{2}\nabla_x \cdot \left\langle |v|^2 vg \right\rangle,
\end{aligned}
\tag{13}
$$

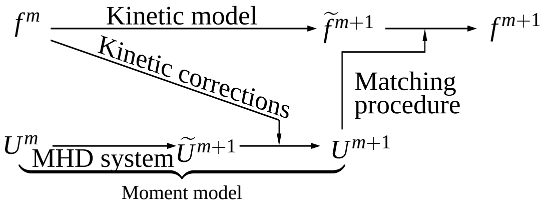
where $\otimes$ is the tensor product, k_B being the Boltzmann constant, the total pressure and energy are defined as

$$
p_{\mathrm{TOT}} = p + \frac{B^2}{2\mu_0}\,, \qquad W_{\mathrm{TOT}} = W + \frac{B^2}{2\mu_0}\,, \qquad p = nk_B T\;, \qquad W = \frac{1}{2}mnu^2 + \frac{1}{\gamma-1}p\,. \tag{14}
$$

These equations are coupled to the induction equation

$$
\frac{\partial B}{\partial t} + \nabla_x \cdot (u \otimes B - B \otimes u) = -\nabla_x \times \left(\frac{\nabla_x \times B}{\mu_0 e n}\right) \times B\,. \tag{15}
$$

Note that the moment model (13 – 15) is very close to the Hall-MHD system but with correction terms as second members for the momentum and energy conservation equations. These kinetic corrections are explained by a distribution function that is not reduced to a Maxwellian. The moment model is derived from the Vlasov equation (10) coupled to the Maxwell-Faraday law (11) without any approximations and should provide the same macroscopic quantity as the one obtained thanks to the Vlasov equation. The guiding strategy performed at each time step is depicted in figure 1. It consists in using the informations provided by the moment model to correct particles properties, so that, the



The distribution function is advanced by means of the kinetic equation providing $\widetilde{f}^{m+1}$. At the same time the macroscopic quantities U^m are advanced thanks to the moment model with kinetic corrections evaluated with f^m and giving U^{m+1}. These quantities are used to perform the matching procedure aimed at correcting particle properties and computing the new distribution function f^{m+1}.

FIGURE 1. Schematic representation of a typical time step for the moment guided method.

moments of the advanced distribution function match the macroscopic quantities computed by the moment model. This correction applied to the particle properties consists, for each cell, in creating or discarding particles, in order to match the first order moment (density), and applying a linear transformation to particle velocity in order to match local mean velocities and pressures. We refer to [5] for a complete description of this correction procedure. This strategy is completely different to that of the most recent methods aimed at reducing the statistical fluctuations, namely the molecular block model DSMC [12] or the Information Preservation DSMC method [7, 15]. The block model exploits the dependance of the statistical error on the gas molecular mass [see 11]. A molecular block account for a large number of particles, and replaces the the conventional simulation particles, with a modified mass and cross section. This method has proven to change the flow Mach number [17] and much work is needed to improve it. The IP-DSMC is an alternative that has received a large interest. In this approach the simulated particles carry additional informations, the preserved quantities, that can be interpreted as an ensemble average from a large set of real molecules represented by the simulation particle. These preserved quantities, initially the velocity [7] referred to as the information velocity, subsequently extended to the density and the temperature [9, 10, 14], are used to evaluate the macroscopic flow field.

However, the derivation of the equations driving the evolution of the preserved quantities relates on some assumptions (a Maxwellian distribution is assumed for each cell in [18]), intuitive formulation or approximations [14, 10]. These methods are more demanding in terms of mermory usage, than the standard (DSMC) ones, and have shown some weakness in describing shock structures [9]. The approach developed here does not suffer any approximation in the moment model derivation and is motivated by the statement that the quantities computed thanks to this model contain less statistical error than the solution of the kinetic equation.

Overview of the numerical methods

As briefly mentioned above, the numerical methods rely on a Particle-In-Cell scheme [2, 8] using macro particles $p \in \mathscr{P}$ defined by their position, velocity and weight $(X_p(t), V_p(t), \omega_p)$ such that $f(x,v,t) = \sum_{p\in\mathscr{P}} \omega_p \delta\left(x - X_p(t)\right)\, \delta\left(v - V_p(t)\right)$, where δ is the Dirac delta function and X_p, V_p satisfy Newton's laws

$$\frac{dX_p(t)}{dt} = V_p(t)\,, \qquad \frac{dV_p(t)}{dt} = \frac{e}{m}\Big(E(X_p) + V_p \times B(X_p)\Big)\,. \tag{16}$$

The particle motion is integrated with a classical leapfrog scheme for which the particle position is computed on integer time steps with velocity on half time steps. A classical Boris Push ([see 2, 8]) is used to integrate the differential equation for the particle velocity. The electromagnetic field is computed on a grid thanks to the definition of the macroscopic quantities defined on each cell according to particle properties. After the computations of the electromagnetic field, the grid quantities are interpolated back onto the particles allowing the Newton's law (16) integration. The projection of macroscopic quantities as well as the interpolation of grid quantities at particle position are achieved thanks to a projection-interpolation scheme such as the *Nearest Grid Point* or the *Cloud In Cell* [2, 8].

The moment model (13 – 15) is discretized thanks to an upwind scheme [6] supplemented with a generalised Lagrangian multiplier [4] to ensure a divergence free magnetic field. The correction terms accumulated from particle properties are finite differenced. For the moment, the Hall term (*i.e.*, the second member) in the induction equation (15) is disregarded and its discretization is reported in future work.

NUMERICAL RESULTS

The efficiency of the moment guided method is illustrated on a simulation carried out for a one dimensional space configuration with three components for both the magnetic field and the velocity. It consists of a "Brio and Wu" shock tube [3] widely used for the validation of numerical schemes designed for the MHD system. The grid is composed of 200 cells. A first order accurate time discretization has been used for all these simulations. The plasma density n and hydrodynamic energy W as well as the B_y magnetic field component are represented in figure 2. The classical

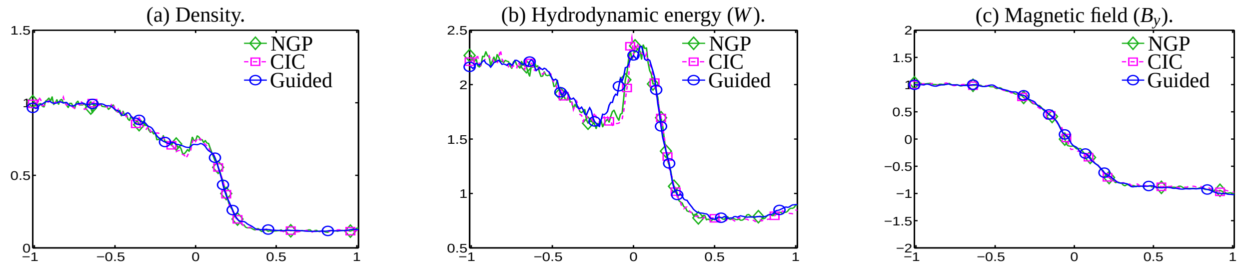


FIGURE 2. Comparisons of the plasma density, hydrodynamic energy and magnetic field component B_y at time $t = 0.25$ s., as computed by the moment guided method, and classical Particle-In-Cell methods using NGP and CIC schemes. For the moment guided method an NGP scheme has been used. The grid is composed of 200 cells, a total of $2 \cdot 10^5$ particles being used.

PIC methods relying on either a NGP or a CIC projection-interpolation scheme are compared with the moment guided method using a NGP scheme. A total of $2 \cdot 10^5$ particles have been used to compute these approximations. The moment model is discretized thanks to a P^0 scheme [6] equivalent to the Rusanov one [13], with a second order MUSCL [16] reconstruction. For the classical PIC method, the magnetic field is computed by the MHD-system discretized by the same space and time schemes, with macroscopic quantities defined from particle properties thanks to the projection scheme. The approximations computed by all three methods are comparable. However, the moment guided method seems to reduce, at least in a small ratio, the numerical noise. The influence of the number of particles as well as the space discretization of the moment model is investigated thanks to additional computations carried out on the same mesh but with more particles ($2 \cdot 10^7$). These results are displayed on figure 3. The approximation computed with a

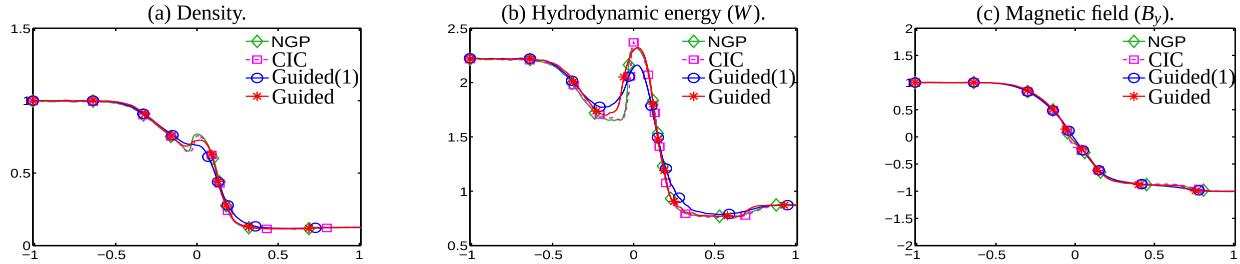


FIGURE 3. Comparisons of the plasma density, hydrodynamic energy and the magnetic field component B_y at time $t = 0.25$ s., as computed by the moment guided method, and classical Particle-In-Cell methods using NGP and CIC schemes. For the moment guided method an NGP has been used with a first (Guided(1)) or second (Guided) order space discretized moment model. The grid is composed of 200 cells with $2 \cdot 10^7$ particles.

first order space disretization is referred to as "Guided(1)" in these plots. The first order scheme gives more diffusive results. This feature is made obvious by observing the curves associated with the first and the second order space discretizations, particularly for the density and hydrodynamic energy plots in the area around the abscissa $x = 0$. The larger number of particles used for this computation allows a better control of the numerical noise as compared to that of the simulation of the figure 2. Note that the approximations provided by the classical PIC method with either the NGP or the CIC schemes are very close.

Finally the numerical noise reduction properties of the moment guided method are investigated. With this aim, a reference solution is computed on the same 200-cells grid with 10^6 particles in a cell. This is ten times more particles as compared to the simulation results of figure 3 for which the numerical noise is already very small. On figure 4 this reference solution is compared with an approximation computed using only 10^3 particles in a cell. Two plots are

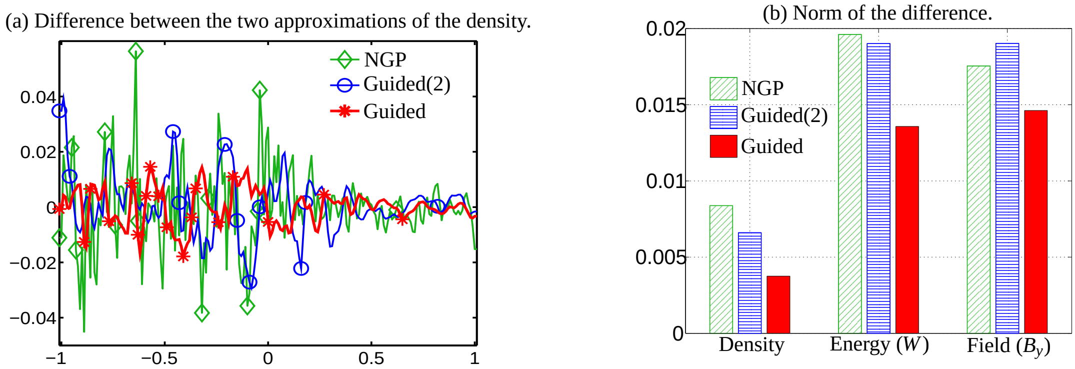


FIGURE 4. Numerical noise of the approximations estimated by the difference between the computations realized with 10^3 and 10^6 particles per cell for the classical PIC methods and the moment guided method with first (Guided) and second order (Guided(2)) space discretizations: (a) Difference as a function of the space variable for the plasma density approximations; (b) l_2-norm of the differences divided by the norm of the variables for the plasma density and energy as well as the magnetic field (B_y component).

displayed, the first one is the difference of the two plasma density approximations as a function of the space variable x, the second one is the l_2-norm of this difference for the plasma density (n) and hydrodynamic energy (W) as well as the magnetic field component B_y. For this later plot, the norm of the differences are divided by the corresponding variables norm, in order to plot dimensionless quantities. The approximations computed thanks to the CIC and NGP schemes being very close, only the NGP results are displayed for the classical PIC method and is compared, on this figure, with the first and second order space discretized moment guided method. The moment guided method with the less diffusive space discretization is observed to produce numerical approximation globally less subject to numerical noise than the standard PIC method (see figure 4(b)). This feature is enforced when a first order space discretization is used, but at the price of larger numerical diffusion altering significantly the approximation quality (in comparison with the approximation computed thanks to standard PIC methods using a large number of particles) as depicted in figure 3(a) and 3(b).

CONCLUSIONS AND PERSPECTIVES

In this paper we have proposed a numerical model for the ionospheric plasma description and its interaction with the earth magnetic field. It consists of a kinetic description of the plasma coupled to the Maxwell-Faraday equation driving the evolution of the magnetic field. The numerical method relies on a Particle-In-Cell method for the kinetic equation discretization and a finite volume scheme for the magnetic field one. A noise reduction method has been tested in this framework. It uses the information carried out by a moment model in order to correct the particles properties and finally reduce the numerical noise characteristic of particles method. The first simulations performed demonstrate some interesting benefits. Future work will be devoted to the extensions of these first results to two dimensional problems with the introduction of more complete collision processes in order to address more accurate and relevant physics. Moreover, as demonstrated in [5], the collisions should increase the efficiency of the moment guided method with respect to the numerical noise reduction.

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